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On aspects of boundary damping for a rectangular plate

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Abstract

In this paper, the vibrations are studied of a rectangular plate with two opposite sides simply supported. To the other two sides of the plate linear springs and dampers are attached. By using the recently developed adapted version of the method of separation of variables the relationship between the plate parameters and the damping rates is obtained analytically.

1. Introduction

Flexible structures, like tall buildings, suspension bridges, and overhead power transmission lines are often subjected to oscillations due to different causes, such as windflows or earthquakes. In some cases such oscillations may cause real trouble. For instance, in 1940 the Tacoma Narrow Bridge collapsed completely because of windflow-induced torsional oscillations of the bridge deck. The collapse of the Tacoma Narrow Bridge clarified the need for developing new methods of designing highway structures to resist wind effects.

Elongated flexible structures such as bridge decks, cables and columns are particularly susceptible to wind loads. Even if they are designed with plenty of strength to resist the static wind load. However, dynamic flow–structure interactions can give rise to trouble for certain ranges of flow speeds. Such troublesome aeroelastic phenomena include flutter, vortex shedding, buffeting and wind-rain-induced vibrations.

According to the Federal Highway Administration [1] flutter occurs when the interaction of a bluff section and the wind create a condition in which small motions of the bridge deck can extract energy from the wind and produce larger motions. This situation is often referred to as negative damping. Flutter produces motions, often in the form of torsional oscillations, that grow exponentially. It is believed nowadays that flutter caused the collapse of the Tacoma Narrow Bridge.

Buffeting is the beating of wind gusts against a structure, causing oscillations. It can lead to long-term fatigue damage and unacceptably large, structural motions. Vortex shedding causes usually high-frequency oscillations with small amplitudes. Galloping oscillations of iced overhead power transmission lines in a windfield are examples of low-frequency vibrations with large amplitudes. These galloping oscillations tend to

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occur at low wind speeds and also give rise to material fatigue. Wind-rain-induced vibrations usually occur at the main cables of cable-stayed bridges. The water changes the aerodynamic behavior of the cable sometimes in such a way that these cables become unstable and start to oscillate. An example was the undesirable vibration of the Erasmus Bridge in Rotterdam during storm and rainy weather just after its opening.

The presence of any of these flow effects on structures are problems that demand attention. Simple models which describe these oscillations can be expressed in initial-boundary value problems for wave equations like in Refs. [2–4], for beam equations like in Refs. [5–16] or for plate equations like in Refs. [17–19]. To suppress the oscillations various types of boundary damping can be applied.

In most cases the literature on plate equations deals with classical boundary conditions representing clamped, simply supported or free edges. Leissa [20] gave a detailed description of 21 distinct cases for rectangular plates which involve all possible combinations of classical boundary conditions. A much smaller number of papers can be found that deal with edges which are elastically restrained against translation and/or rotation or with other nonclassical boundary conditions (see, for instance, Refs. [18,19,21]). In this paper, boundary damping (such as damping which is proportional to vertical and angular velocities) will be included in the boundary conditions. Compared to the existing literature these types of boundary damping for plate equations seem to be not widely studied before.

In this paper, a plate equation will be used to study for instance the flow-induced oscillations of a suspension bridge. The deck of the bridge is modeled as a rectangular plate, and the cables are modeled as linear springs densely attached to two opposite edges of the plate (Figs. 1 and 2). Such a plate configuration can be described by the following initial-boundary value problem:

$$\rho u_{\bar{t}\bar{t}} + D(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) = \varepsilon \bar{f}(x, y, \bar{t}, u, u_{\bar{t}}), \quad 0<x<l, \ 0<y<d, \ t>0, \tag{1}$$

$$u(x, y, 0) = u_0(x, y), \quad u_{\bar{t}}(x, y, 0) = u_1(x, y), \quad 0<x<l, \ 0<y<d, \tag{2}$$

$$u(0, y, \bar{t}) = u(l, y, \bar{t}) = u_{xx}(0, y, \bar{t}) = u_{xx}(l, y, \bar{t}) = 0, \quad 0<y<d, \tag{3}$$

$$D(u_{yyy} + (2 - v)u_{xxy}) = -\bar{p}^2 u - \varepsilon\bar{\alpha}_1 u_{\bar{t}} \quad \text{for } y = 0, \ 0<x<l, \tag{4}$$

$$D(u_{yyy} + (2 - v)u_{xxy}) = \bar{p}^2 u + \varepsilon\bar{\alpha}_1 u_{\bar{t}} \quad \text{for } y = d, \ 0<x<l, \tag{5}$$

$$u_{yy} + v u_{xx} = -\varepsilon\bar{\delta} u_{\bar{t}y} \quad \text{for } y = 0, \ 0<x<l, \tag{6}$$

$$u_{yy} + v u_{xx} = \varepsilon\bar{\delta} u_{\bar{t}y} \quad \text{for } y = d, \ 0<x<l, \tag{7}$$

where $u(x, y, \bar{t})$ is the vertical displacement of the plate in z-direction, $\bar{t}$ is time, l and d are the length and the width of the plate, respectively, ε is a small, positive dimensionless parameter, $\bar{\alpha}_1$ and $\bar{\delta}$ are positive damping

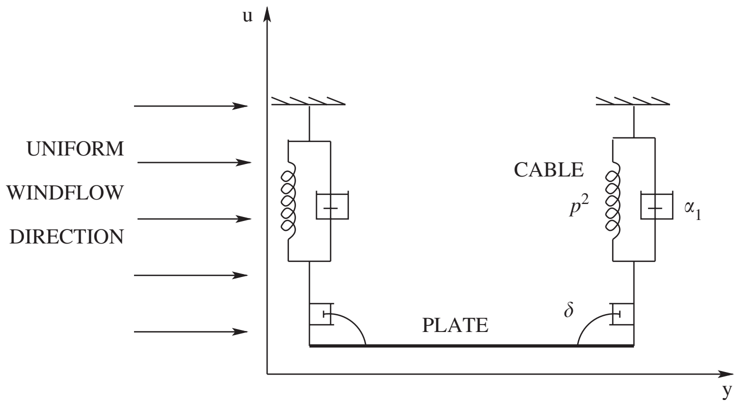

Fig. 1. A schematic representation of the bridge-cable-system at a cross-section in the (u, y) plane for a fixed $x = x_0$ with $0<x_0<l$.

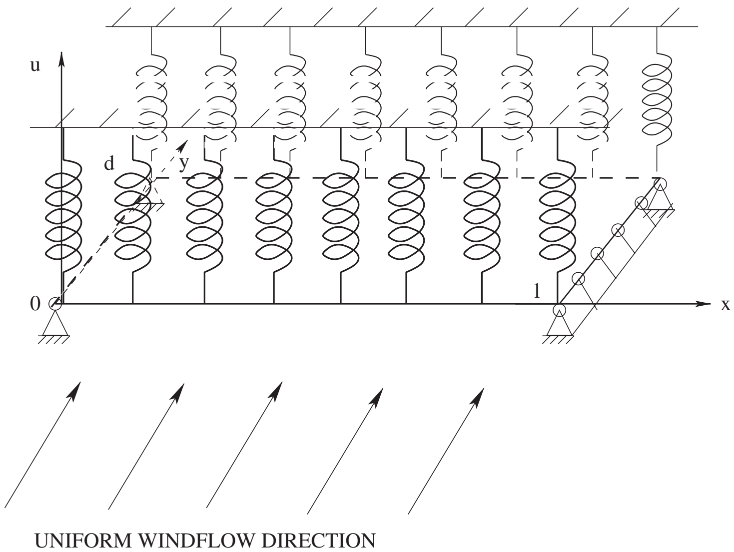

Fig. 2. A model of a suspension bridge (dampers are not depicted).

constants, $\bar{p}^2$ represents the linear restoring forces of the springs, $D = Eh^3/12(1-v^2)$ is the flexural rigidity, E is Young's modulus, v is the Poisson's ratio with $0<v<0.5$, ρ is the mass density per unit area of the plate surface, h is the thickness of the plate, and $\bar{f}(x,y,\bar{t},u,u_{\bar{t}})$ is a (usually) nonlinear, aerodynamical force (an example of which is explicitly given in Ref. [17]). The initial displacement and the initial velocity of the plate in z-direction are given by $u_0(x,y)$ and $u_1(x,y)$, respectively. The boundary conditions (4)–(7) are nonclassical ones. These seem not to have been studied in the literature before. These boundary conditions describe two types of dampers that are attached to the edges of the plate as can be seen from Fig. 1. These dampers model as damping which is proportional to the vertical velocity (see Refs. [4,5]) and damping which is proportional to the angular velocity (see also Refs.[6,22]).

2. Analysis of the linearized problem

Using simple transformation of time $t = \sqrt{(D/\rho)}\bar{t}$ initial-boundary value problem (1)–(7) can be simplified and be written in the following form:

$$
\begin{aligned}
&u_{tt} + u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \varepsilon f(x,y,t,u,u_t), \quad 0<x<l,\ 0<y<d,\ t>0,\\
&u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \quad 0<x<l,\ 0<y<d,\\
&u(0,y,t) = u(l,y,t) = u_{xx}(0,y,t) = u_{xx}(l,y,t) = 0, \quad 0<y<d,\\
&u_{yyy} + (2-v)u_{xxy} = -p^2 u - \varepsilon\alpha_1 u_t \quad \text{for } y=0,\ 0<x<l,\\
&u_{yyy} + (2-v)u_{xxy} = p^2 u + \varepsilon\alpha_1 u_t \quad \text{for } y=d,\ 0<x<l,\\
&u_{yy} + v u_{xx} = -\varepsilon\delta u_{ty} \quad \text{for } y=0,\ 0<x<l,\\
&u_{yy} + v u_{xx} = \varepsilon\delta u_{ty} \quad \text{for } y=d,\ 0<x<l,
\end{aligned} \tag{8}
$$

where $f = \bar{f}/D$, $p^2 = \bar{p}^2/D$, $\alpha_1 = \bar{\alpha}_1/\sqrt{\rho D}$, and $\delta = \bar{\delta}\sqrt{D/\rho}$. As was mentioned before, one of the simplest models of suspension bridges can be given by a rectangular plate with two opposite edges simply supported and linear springs densely attached to the two other edges (see Fig. 2). In this section such a plate will be considered in a wind-flow. We will linearize the initial-boundary problem (8), and will deal only with the terms of $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$, because these terms have the most significant influence on the stability of the system. The term $\varepsilon a u_t$ will be considered because it is known that it brings instability in the system (due to instability criterion of Den Hartog [23]). After linearization the initial-boundary problem (8) transforms to

$$u_{tt} + u_{xxxx} + 2u_{xxyy} + u_{yyyy} = a\varepsilon u_t, \quad 0<x<l, \ 0<y<d, \ t>0, \tag{9}$$

$$u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \quad 0<x<l, \ 0<y<d, \tag{10}$$

$$u(0,y,t) = u(l,y,t) = u_{xx}(0,y,t) = u_{xx}(l,y,t) = 0, \quad 0<y<d, \tag{11}$$

$$u_{yyy} + (2-\nu)u_{xxy} = -p^2 u - \varepsilon\alpha_1 u_t \quad \text{for } y=0, \ 0<x<l, \tag{12}$$

$$u_{yyy} + (2-\nu)u_{xxy} = p^2 u + \varepsilon\alpha_1 u_t \quad \text{for } y=d, \ 0<x<l, \tag{13}$$

$$u_{yy} + \nu u_{xx} = -\varepsilon\delta u_{ty} \quad \text{for } y=0, \ 0<x<l, \tag{14}$$

$$u_{yy} + \nu u_{xx} = \varepsilon\delta u_{ty} \quad \text{for } y=d, \ 0<x<l, \tag{15}$$

where a is positive aerodynamical constant. Let us make the following transformation $u(x,y,t) = e^{(\varepsilon a t/2)} v(x,y,t)$. Then system (9)–(15) can be rewritten as

$$v_{tt} + v_{xxxx} + 2v_{xxyy} + v_{yyyy} = \mathcal{O}(\varepsilon^2), \quad 0<x<l, \ 0<y<d, \ t>0, \tag{16}$$

$$v(x,y,0) = v_0(x,y), \quad v_t(x,y,0) = v_1(x,y), \quad 0<x<l, \ 0<y<d, \tag{17}$$

$$v(0,y,t) = v(l,y,t) = v_{xx}(0,y,t) = v_{xx}(l,y,t) = 0, \quad 0<y<d, \tag{18}$$

$$v_{yyy} + (2-\nu)v_{xxy} = -p^2 v - \varepsilon\alpha_1 v_t + \mathcal{O}(\varepsilon^2), \quad y=0, \ 0<x<l, \tag{19}$$

$$v_{yyy} + (2-\nu)v_{xxy} = p^2 v + \varepsilon\alpha_1 v_t + \mathcal{O}(\varepsilon^2), \quad y=d, \ 0<x<l, \tag{20}$$

$$v_{yy} + \nu v_{xx} = -\varepsilon\delta v_{ty} + \mathcal{O}(\varepsilon^2), \quad y=0, \ 0<x<l, \tag{21}$$

$$v_{yy} + \nu v_{xx} = \varepsilon\delta v_{ty} + \mathcal{O}(\varepsilon^2), \quad y=d, \ 0<x<l. \tag{22}$$

Now, the $\mathcal{O}(\varepsilon^2)$ terms are neglected and the problem (16)–(22) will be studied further. It will be shown for positive values of δ and α_1 that the energy related to v decreases for increasing times. For this purpose an energy of system (16)–(22) will be written

$$E(t) = U + P_1 + P_2, \tag{23}$$

where U is the kinetic energy of the plate, P_1 is the potential energy of the plate and P_2 is the potential energy due to linear springs (which are densely attached to the two edges at $y=0$ and $y=d$), that is

$$U = \frac{1}{2}\int_0^t \int_0^d v_t^2, \quad P_1 = \int_0^t \int_0^d \left(\frac{1}{2}v_{xx}^2 + \frac{1}{2}v_{yy}^2 + (1-\nu)v_{xy}^2 + \nu v_{xx}v_{yy}\right) \mathrm{d}x\,\mathrm{d}t,$$

$$P_2 = \frac{1}{2}p^2 \int_0^t \int_0^l (v^2(x,0,t) + v^2(x,d,t))\,\mathrm{d}x\,\mathrm{d}t.$$

It is well known that the damping occurs in a system if the first time derivative from energy is nonpositive. In this case it follows that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\varepsilon\alpha_1 \int_0^l (v_t^2(x,0,t) + v_t^2(x,d,t))\,\mathrm{d}x - \varepsilon\delta \int_0^l (v_{ty}^2(x,0,t) + v_{ty}^2(x,d,t))\,\mathrm{d}x \leqslant 0. \tag{24}$$

From the last expression it can clearly be seen that the first-order time derivative of the energy is nonpositive, so damping occurs in the system. The adapted version of the method of separation of variables [21,22] can be used to find nontrivial solutions of Eqs. (16)–(22) in a form $T(t)X(x)Y(y)$. Substituting this expression for

$v(x,y,t)$ into Eq. (16) and dividing by $X(x)Y(y)T(t)$ we obtain the following:

$$\frac{\mathrm{d}^2T}{T\,\mathrm{d}t^2}+\frac{\ddot{\ddot{X}}}{X}+2\frac{\ddot{X}}{X}\frac{Y''}{Y}+\frac{Y''''}{Y}=0, \tag{25}$$

where $\cdot$ stands for $\mathrm{d}/\mathrm{d}x$ and $'$ stands for $\mathrm{d}/\mathrm{d}y$. By using the method of separation of variables follows from the boundary conditions (25) that $(1/T)(\mathrm{d}T/\mathrm{d}t)=\mu$, and from Eq. (25) that

$$\frac{\mathrm{d}T}{T\,\mathrm{d}t}=\mu, \implies \frac{\mathrm{d}^2T}{T\,\mathrm{d}t^2}=\mu^2, \quad \frac{\ddot{X}}{X}=-\beta \implies \frac{\ddot{\ddot{X}}}{X}=\beta^2, \tag{26}$$

where μ and β are complex-valued separation constants. And so, it follows from Eqs. (16)–(22) and Eq. (25) that

$$Y''''-2\beta Y''+(\beta^2+\mu^2)Y=0, \quad 0<y<d, \tag{27}$$

$$Y''-v\beta Y=-\varepsilon\delta\mu Y', \quad y=0, \tag{28}$$

$$Y''-v\beta Y=\varepsilon\delta\mu Y', \quad y=d, \tag{29}$$

$$Y'''-(2-v)\beta Y'=-(p^2+\varepsilon\alpha_1\mu)Y, \quad y=0, \tag{30}$$

$$Y'''-(2-v)\beta Y'=(p^2+\varepsilon\alpha_1\mu)Y, \quad y=d, \tag{31}$$

$$X=\ddot{X}=0, \quad x=0 \text{ and } x=l. \tag{32}$$

The characteristic equation for the differential equation (27) is

$$k^4-2\beta k^2+\beta^2=-\mu^2, \tag{33}$$

where from Eq. (25) and from boundary condition (32) it follows that $\beta=\beta_n=(n\pi/l)^2$ with $n\in\mathbb{Z}^+$. It is assumed that k and μ can be expanded in formal power series in ε, so $k=k_0+\varepsilon k_1+\cdots$ and $\mu=\mu_0+\varepsilon\mu_{01}+\cdots$. Collecting equal powers of ε in Eq. (33) we obtain

$$\mathcal{O}(1): \quad k_0^4-2\beta_n k_0^2+\beta_n^2=-\mu_0^2, \tag{34}$$

$$\mathcal{O}(\varepsilon): \quad 4k_0k_1(k_0^2-\beta_n)=-2\mu_0\mu_{01}. \tag{35}$$

For $\varepsilon=0$ this problem completely coincides with the linear problem which has been studied in Ref. [17]. By putting $-\mu_0^2=\alpha$ it follows from Ref. [17] that α is always real and positive, so $\mu_0=\pm\mathrm{i}\gamma$, where $\gamma=\sqrt{\alpha}$. In Ref. [17] it has been shown that there are three cases that have to be considered in Eq. (34): $\gamma>\beta_n$, $0<\gamma<\beta_n$ and $\gamma=-\beta_n$.

2.1. The case $\gamma>\beta_n$

The solutions of the characteristic equation (34) in this case are $k_0=\pm k_{01}$ with $k_{01}=\sqrt{\gamma+\beta_n}$ or $k_0=\pm k_{02}$ with $k_{02}=\mathrm{i}\sqrt{\gamma-\beta_n}$. The corresponding solutions k_1 of the $\mathcal{O}(\varepsilon)$ problem (35) will be then $k_1=-\mathrm{i}\mu_{01}/2k_0$. To simplify the calculations the following is introduced:

$$k_{11}=\frac{\mu_2}{2\sqrt{\gamma+\beta_n}}-\mathrm{i}\frac{\mu_1}{2\sqrt{\gamma+\beta_n}}, \quad k_{12}=\frac{\mu_1}{2\sqrt{\gamma-\beta_n}}+\mathrm{i}\frac{\mu_2}{2\sqrt{\gamma-\beta_n}}, \tag{36}$$

where $\mu_{01}=\mu_1+\mathrm{i}\mu_2$. The general solution of Eq. (27) can then be written in the form

$$Y(y)=C_1\cosh(m_1y)+C_2\sinh(m_1y)+C_3\cosh(m_2y)+C_4\sinh(m_2y), \tag{37}$$

where C_1, C_2, C_3, C_4 are constants of integration and $m_1=k_{01}+\varepsilon k_{11}+\mathcal{O}(\varepsilon^2)$, $m_2=k_{02}+\varepsilon k_{12}+\mathcal{O}(\varepsilon^2)$. By substituting Eq. (37) into the four boundary conditions (28)–(31) a system of four equations for C_1, C_2, C_3, and C_4 is obtained. To find nontrivial solutions for $Y(y)$ the determinant of the corresponding coefficient matrix should be set equal to zero, that is,

$$\begin{vmatrix} m_1^2 - v\beta_n & 0 & m_2^2 + v\beta_n & 0 \\ j_1 & s_1 & j_2 & s_2 \\ p^2 + \varepsilon\alpha_1\mu & m_1(m_1^2 - v_1\beta_n) & p^2 + \varepsilon\alpha_1\mu & m_2(m_2^2 - v_1\beta_n) \\ f_1 & h_1 & f_2 & h_2 \end{vmatrix} = 0, \quad (38)$$

where $j_i = (m_i^2 - v\beta_n)\cosh(m_i d) - \varepsilon\delta\mu m_i \sinh(m_i d)$, $s_i = (m_i^2 - v\beta_n)\sinh(m_i d) - \varepsilon\delta\mu m_i \sinh(m_i d)$, $f_i = m_i(m_i^2 - v_1\beta_n)\sinh(m_i d) - (p^2 + \varepsilon\alpha_1\mu)\cosh(m_i d)$, $h_i = m_i(m_i^2 - v_1\beta_n)\cosh(m_i d) - (p^2 + \varepsilon\alpha_1\mu)\sinh(m_i d)$ for $i = 1, 2$. Functions j_i, s_i, f_i and h_i are functions of m_1 and m_2 which we expand in a power series in ε. Then Eq. (38) can be expanded into a power series in ε. Then the $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon), \ldots$ terms are collected and set to equal zero. The result of $\mathcal{O}(1)$ problem sets equal to zero is given in Ref. [17]. Now we will solve $\mathcal{O}(\varepsilon)$ problem. We are interested only in the damping rates which are determined by μ ($\mathrm{d}T/T\,\mathrm{d}t = \mu = \mu_0 + \varepsilon\mu_{01} + \cdots$). It is known that μ_0 is purely imaginary (see Ref. [17])). For the damping rates only the real part of μ_{01} is important, that is, μ_1. By using the formula manipulation package Maple the following expression for μ_1 can be obtained:

$$\begin{aligned} \mu_1 = [&2p^2\delta\gamma B_0(B_{10} - 2\beta_n B_{11} + 2v_2) + 2p^2\delta\gamma B_0(B_{10} + 2\beta_n B_{11} + 2(2B_1 + v_2))\cosh(\gamma_1 d)\cos(\gamma_2 d) \\ &- 2\gamma\gamma_1 B_2(-\alpha_1(\gamma_1^2 - v_1\beta_n)(\gamma_2^2 + v\beta_n) + \delta\{-p^4 + \gamma_2^2(\gamma_1^2 - v\beta_n)(\gamma_2^2 + v_1\beta_n)\})\cosh(\gamma_1 d)\sin(\gamma_2 d) \\ &- 2\gamma\gamma_2 B_2(\alpha_1(\gamma_1^2 - v\beta_n)(\gamma_2^2 + v_1\beta_n) + \delta\{p^4 + \gamma_1^2(\gamma_1^2 - v_1\beta_n)(\gamma_2^2 + v\beta_n)\})\sinh(\gamma_1 d)\cos(\gamma_2 d) \\ &- 2p^2\gamma(\alpha_1 B_2^2 + \delta\{v\beta_n B_{10} + B_7 B_{11} - v_1\beta_n B_1\})\sinh(\gamma_1 d)\sin(\gamma_2 d)]/(b_0 + b_1 + b_2 + b_3 + b_4), \end{aligned}$$

where

$$\gamma_1 = \sqrt{\gamma + \beta_n}, \quad \gamma_2 = \sqrt{\gamma - \beta_n}, \quad v_1 = 2 - v, \quad v_2 = vv_1\beta_n^2, \quad v_3 = (5v^2 - 10v + 12)\beta_n^2, \quad v_4 = v^2 v_1^2 \beta_n^4,$$

$$v_5 = (3v - 4)\beta_n, \quad v_6 = v^2\beta_n^2, \quad v_7 = v_1^2\beta_n^2, \quad v_8 = 2(3 - v), \quad v_9 = 2(5v - 7)\beta_n, \quad v_{10} = (3v^2 - 2v - 2)\beta_n^2,$$

$$v_{11} = 8 - 7v, \quad v_{12} = 4(1 - v), \quad B_0 = \gamma_1\gamma_2, \quad B_1 = B_0^2, \quad B_2 = \gamma_1^2 + \gamma_2^2, \quad B_3 = \gamma_1^4 - \gamma_2^4, \quad B_4 = \gamma_1^6 + \gamma_2^6,$$

$$B_5 = \gamma_1^6 - \gamma_2^6, \quad B_6 = 3\gamma_1^2\gamma_2^2 - v_2, \quad B_7 = B_1 - v_2, \quad B_8 = v_1\gamma_1^4 + v\gamma_2^4, \quad B_9 = v\gamma_1^4 + v_1\gamma_2^4, \quad B_{10} = \gamma_1^4 + \gamma_2^4,$$

$$B_{11} = \gamma_1^2 - \gamma_2^2, \quad b_0 = -\frac{v_2 B_4 + 2\beta_n B_3 B_6 + (5B_1^2 - v_3 B_1 + v_4)B_2}{B_0},$$

$$b_1 = \left(p^2 d\frac{v\beta_n B_4 + B_3 B_7 + v_5 B_1 B_2}{2B_0} + b_0\right)\cosh(\gamma_1 d)\cos(\gamma_2 d),$$

$$\begin{aligned} b_2 = \Big\{&-2p^2\frac{v_2(5\gamma_1^2 + \gamma_2^2) - 3B_1(3\gamma_1^2 + \gamma_2^2) - 2v_9 B_1 + v_1\beta_n(2\gamma_1^4 + \gamma_2^4) - \gamma_1^4(2\gamma_2^2 + 7v\beta_n)}{2\gamma_1} \\ &- d\frac{p^4 B_2^2 + v_6 B_5 + (B_1^2 - v_7 B_1 + v_4)(3\gamma_1^2 - \gamma_2^2) + 2\beta_n B_7(B_8 - v_8 B_1) - 2v_2 B_1(4\gamma_1^2 - 3\gamma_2^2)}{2\gamma_1} \\ &-d\frac{-2v_6\gamma_1^4(\gamma_2^2 + 2v_1\beta_n) + 2v\beta_n\gamma_1^6(2\gamma_2^2 + v_1\beta_n)}{2\gamma_1}\Big\}\cosh(\gamma_1 d)\sin(\gamma_2 d), \end{aligned}$$

$$\begin{aligned} b_3 = \Big\{&-2p^2\frac{-v_2(\gamma_1^2 + 5\gamma_2^2) + 3B_1(\gamma_1^2 + 3\gamma_2^2) + 2v_9 B_1 + v_1\beta_n(\gamma_1^4 + 2\gamma_2^4) + \gamma_2^4(2\gamma_1^2 - 7v\beta_n)}{2\gamma_2} \\ &- d\frac{p^4 B_2^2 + v_6 B_5 + (B_1^2 - v_7 B_1 + v_4)(\gamma_1^2 - 3\gamma_2^2) + 2\beta_n B_7(B_9 - v_8 B_1) - 2v_2 B_1(3\gamma_1^2 - 4\gamma_2^2)}{2\gamma_2} \\ &-d\frac{2v_6\gamma_2^4(\gamma_1^2 - 2v_1\beta_n) + 2v\beta_n\gamma_2^6(2\gamma_1^2 - v_1\beta_n)}{2\gamma_2}\Big\}\sinh(\gamma_1 d)\cos(\gamma_2 d), \end{aligned}$$

$$b_4 = 2(p^2 d(B_2 B_7 + \beta_n B_3) + 2p^4 B_2 - v\beta_n B_4 - (B_1 + v_{10})B_3 + \beta_n(v_{11}B_1 - v_{12}v_2)B_2)\sinh(\gamma_1 d)\sin(\gamma_2 d).$$

2.2. The case $0<\gamma<\beta_n$

The solutions of the characteristic equation (34) in this case are $k_0 = \pm k_{01}$ with $k_{01} = \sqrt{\gamma+\beta_n}$ or $k_0 = \pm k_{02}$ with $k_{02} = \mathrm{i}\sqrt{\gamma-\beta_n}$. The corresponding solutions k_1 of the $\mathcal{O}(\varepsilon)$ problem (35) will be then $k_1 = -2\mu_0\mu_{01}/(4k_0(k_0^2-\beta_n))$. To simplify the calculations the following is introduced:

$$k_{11} = \frac{\mu_2}{2\sqrt{\gamma+\beta_n}} - \mathrm{i}\frac{\mu_1}{2\sqrt{\gamma+\beta_n}}, \quad k_{12} = -\frac{\mu_2}{2\sqrt{\gamma-\beta_n}} + \mathrm{i}\frac{\mu_1}{2\sqrt{\gamma-\beta_n}}, \tag{39}$$

where $\mu_{01} = \mu_1 + \mathrm{i}\mu_2$. The general solution of Eq. (27) can be written in the form (37). By substituting Eq. (37) into the four boundary conditions (28)–(31) a system of four equations for C_1, C_2, C_3, and C_4 is obtained. To find nontrivial solutions for $Y(y)$ the determinant of the corresponding coefficient matrix should be set equal to zero. As in the previous case we are interested in the damping rate μ. Eq. (38) should be expanded in a power series of ε. The $\mathcal{O}(1)$ problem has already been solved in Ref. [17]. As in case when $\gamma>\beta_n$ we will deal with $\mathcal{O}(\varepsilon)$ problem. The real part of μ_{01}, that is, μ_1 describes damping. With the help of Maple the following expression for determination of μ_1 is obtained:

$$\begin{aligned}\mu_1 = {} & [-2p^2\delta\gamma P_0(P_{10} - 2\beta_n P_2 + 2\sigma_2) - 2p^2\delta\gamma P_0(P_{10} + 2\beta_n P_2 - 2(2P_1+\sigma_2))\cosh(\gamma_1 d)\cosh(\gamma_3 d) \\ & - 2\gamma\gamma_1 P_6(-\alpha_1 P_{11}P_{12} + \delta(p^4 - \gamma_3^2 P_{13}P_{14}))\cosh(\gamma_1 d)\sinh(\gamma_3 d) - 2\gamma\gamma_3 P_6(-\alpha_1 P_{13}P_{14} \\ & + \delta(p^4 - \gamma_1^2 P_{11}P_{12}))\sinh(\gamma_1 d)\cosh(\gamma_3 d) - 2p^2\gamma(-\alpha_1 P - 2^2 + \delta(-v\beta_n P_{10} + P - 2P_7 \\ & - 2v_1\beta_n P_1))\sinh(\gamma_1 d)\sinh(\gamma_3 d)]/(a_0 + a_1 + a_2 + a_3 + a_4),\end{aligned} \tag{40}$$

where

$$\gamma_3 = \sqrt{\beta_n - \gamma}, \quad \sigma_2 = vv_1\beta_n^2, \quad \sigma_3 = (5v^2 - 10v + 12)\beta_n^2, \quad \sigma_4 = v^2v_1^2\beta_n^4, \quad \sigma_5 = (4-3v)\beta_n, \quad \sigma_6 = v^2\beta_n^2,$$

$$\sigma_7 = v_1^2\beta_n^2, \quad \sigma_8 = 2(3-v), \quad \sigma_9 = 2(5v-)\beta_n, \quad \sigma_{10} = (3v^2 - 2v - 2)\beta_n^2, \quad \sigma_{11} = 8 - 7v, \quad \sigma_{12} = 4(1-v),$$

$$\begin{aligned}&P_0 = \gamma_1\gamma_3, \quad P_1 = P_0^2, \quad P_2 = \gamma_1^2 + \gamma_3^2, \quad P_3 = \gamma_1^4 - \gamma_3^4, \\ &P_4 = \gamma_1^6 + \gamma_3^6, \quad P_5 = \gamma_1^6 - \gamma_3^6, \quad P_2 = \gamma_1^2 - \gamma_3^2,\end{aligned}$$

$$\begin{aligned}&P_7 = P_1 + \sigma_2, \quad P_8 = v_1\gamma_1^4 + v\gamma_3^4, \quad P_9 = v\gamma_1^4 + v_1\gamma_3^4, \\ &P_{10} = \gamma_1^4 + \gamma_3^4, \quad P_{11} = \gamma_1^2 - v_1\beta_n, \quad P_{12} = \gamma_2^2 - v\beta_n,\end{aligned}$$

$$P_{13} = \gamma_1^2 - v\beta_n, \quad P_{14} = \gamma_2^2 - v_1\beta_n, \quad a_0 = -\frac{1}{P_0}(\sigma_2 P_5 - 2\beta_n P_3(3P_1 + \sigma_2) + (5P_1^2 + \sigma_3 P_1 + \sigma_4)P_6),$$

$$a_1 = \left(p^2 d\frac{v\beta_n P_5 - P_3P_7 + \sigma_5 P_1 P_6}{P_0} - a_0\right)\cosh(\gamma_1 d)\cosh(\gamma_3 d),$$

$$\begin{aligned}a_2 = {} & \left\{ d\frac{p^4P_6^2 - \sigma_6 P_4 + (P_1^2 + \sigma_7 P_1 + \sigma_4)(3\gamma_1^2 + \gamma_3^2) - 2\beta_n P_7(P_8 + \sigma_8 P_1) - 2v\beta_n\gamma_1^6(2\gamma_3^2 - v_1\beta_n)}{2\gamma_1} \right. \\ & + d\frac{2\sigma_6\gamma_1^4(\gamma_3^2 - 2v_1\beta_n) + 2\sigma_2 P_1(4\gamma_1^2 + 3\gamma_3^2)}{2\gamma_1} + p^2\frac{\sigma_2(5\gamma_1^2 - \gamma_3^2) + 3P_1(3\gamma_1^2 - \gamma_3^2) + \sigma_9 P_1}{\gamma_1} \\ & \left. + p^2\frac{v_1\beta_n(2\gamma_1^4 + \gamma_3^4) - \gamma_1^4(2\gamma_1^2 + 7v\beta_n)}{\gamma_1} \right\}\cosh(\gamma_1 d)\sinh(\gamma_3 d),\end{aligned}$$

$$a_3 = \left\{ d\frac{p^4P_6^2 + \sigma_6P_4 + (P_1^2 + \sigma_7P_1 + \sigma_4)(\gamma_1^2 + 3\gamma_3^2) - 2\beta_nP_7(P_9 + \sigma_8P_1) - 2v\beta_n\gamma_3^6(2\gamma_1^2 - v_1\beta_n)}{2\gamma_3} \right.$$
$$\times d\frac{2\sigma_6\gamma_3^4(\gamma_1^2 - 2v_1\beta_n) + 2\sigma_2P_1(3\gamma_1^2 + 4\gamma_3^2)}{2\gamma_1} + p^2\frac{-\sigma_2(\gamma_1^2 - 5\gamma_3^2) - 3P_1(\gamma_1^2 - 3\gamma_3^2) + \sigma_9P_1}{\gamma_3}$$
$$\left. + p^2\frac{v_1\beta_n(\gamma_1^4 + 2\gamma_3^4) - \gamma_3^4(2\gamma_3^2 + 7v\beta_n)}{\gamma_3} \right\} \sinh(\gamma_1d)\cosh(\gamma_3d),$$

$$a_4 = 2(p^2d(P_6P_7 - 2\beta_nP_3) - v\beta_nP_5 + (P_1 - \sigma_{10})P_3$$
$$+ 2\beta_n(\sigma_{11}P_1 - \sigma_{12}\sigma_2)P_6 + 2p^4P_6)\sinh(\gamma_1d)\sinh(\gamma_3d).$$

In Table 1 some numerical values of μ_1 are given for $l = 10$, $d = 1$, and $p^2 = 1$, and both cases $\gamma > \beta_n$ and $0 < \gamma < \beta_n$. It should be observed that the real part of the eigenvalues μ is equal to $\varepsilon\mu_1 + 0(\varepsilon^2)$.

2.3. The case $\gamma = -\beta_n$

This case is equivalent to the case $\mu^2 = -\beta_n^2$, and the characteristic equation (35) becomes

$$k^2(k^2 - 2\beta_n) = 0. \tag{41}$$

The solutions of the characteristic equation (41) have the following form:

$$k_{1,2} = 0, \quad k_{3,4} = \pm\sqrt{2\beta_n}.$$

The solution of differential equation (27) is then given by

$$Y(y) = S_1 + S_2y + S_3\cosh\left(\sqrt{2\beta_n}y\right) + S_4\sinh\left(\sqrt{2\beta_n}y\right), \tag{42}$$

Table 1
Approximations of the smallest values of μ_1 (in absolute value) for both cases $\gamma > \beta_n$ and $0 < \gamma < \beta_n$

n	$v = 0.3$				
1	−0.855297	−22.341728	−433.570945	−1938.959555	−5310.646486
2	−0.365661	−51.177951	−472.567353	−1990.159190	−5373.548805
3	−0.248917	−74.642400	−539.166330	−2075.468321	−5478.159107
4	−0.302857	−71.725997	−634.933345	−2194.710712	−5624.096416
5	−0.456099	−66.622493	−760.536090	−2374.380172	−5810.748641
6	−0.666205	−82.120460	−914.621958	−2532.431739	−6037.205592
7	−0.905629	−113.319536	−1093.021721	−2748.079258	−6302.184704
8	−0.950153	−122.094399	−1370.698084	−2991.644196	−6603.960319
9	−1.207062	−130.647248	−1492.106746	−3259.485373	−6940.366182
10	−1.463350	−171.436777	−1693.489284	−3547.836399	−7308.457540
n	$v = 0.4999$				
1	−1.065523	−21.565956	−431.037702	−1934.867307	−5305.084414
2	−0.606753	−50.356860	−462.559117	−1973.857451	−5351.354872
3	−0.441742	−74.235099	−517.330501	−2039.067654	−5428.433226
4	−0.446651	−62.423830	−598.595286	−2130.786084	−5536.245554
5	−0.580960	−63.402597	−710.702024	−2249.319256	−5674.660087
6	−0.734439	−66.858459	−858.991944	−2394.910204	−5843.460466
7	−0.983150	−70.724651	−1049.759653	−2567.652582	−6042.319116
8	−1.234722	−74.381885	−1290.355565	−2767.414212	−6270.767441
9	−1.488899	−77.635734	−1589.504836	−2993.778782	−6528.171732
10	−1.751121	−80.438284	−1957.918596	−3246.015997	−6813.713905

where S_1, S_2, S_3 and S_4 are constants of integration. As in the previous two cases the following determinant can similarly be obtained when we look for nontrivial solutions of the boundary value problem for $Y(y)$ (where $Y(y)$ is given by Eq. (42)). Like in the previous two cases by using boundary conditions (28)–(31) a system of the four equations for S_1, S_2, S_3, and S_4 is obtained. This system has nontrivial solution when the determinant of the coefficient matrix for the unknown quantities $S_i = 0,\ i = 1, 2, 3, 4$ is equal to zero. In this case the determinant has the following form:

$$\begin{vmatrix} -\nu\beta_n & \varepsilon\delta\mu & b^2-\nu\beta_n & \varepsilon\delta\mu b \\ -\nu\beta_n & -(\varepsilon\delta\mu+\nu\beta_n d) & (b^2-\nu\beta_n)\cosh(bd)-\varepsilon\delta\mu\sinh(bd) & (b^2-\nu\beta_n)\sinh(bd)-\varepsilon\delta\mu\cosh(bd) \\ \rho & -\nu\beta_n & \rho & b(b^2-\nu_1\beta_n) \\ -\rho & -(\nu\beta_n+\rho d) & b(b^2-\nu_1\beta_n)\sinh(bd)-\rho\cosh(bd) & b(b^2-\nu_1\beta_n)\cosh(bd)-\rho\sinh(bd) \end{vmatrix} = 0, \quad (43)$$

where $b = \sqrt{2\beta_n}, \rho = p^2 + \varepsilon\alpha_1\mu$. Solutions of Eq. (43) do not always exist. The existence strongly depends on the values of the geometrical and the physical parameters of plate and springs. For example for $l = 10, d = 1, p^2 = 1, \nu = 0.3, \alpha_1 = 1$ only a finite number of solutions exist, that is, only finite number of modes oscillate with a shape-form given by Eq. (42). For other values of the parameters it may turns out that Eq. (43) has no solutions at all (so only trivial solutions exist in this case).

3. Conclusions

In this paper, the damped vibrations of a rectangular plate with nonclassical boundary conditions have been studied. These combinations of boundary conditions seems to be not considered in the literature before. These rectangular plates may serve as simple models for a suspension bridge. For the rectangular plate the relationship between the plate parameters, the frequencies and the damping rates have been obtained by using an adapted version of the method of separation of variables (see Refs. [21,22]). This relationship has been obtained analytically, and numerically approximations of the damping rates are given.

In this paper the analysis has been restricted to a linear model equation. Although, for more realistic situations nonlinearities and torsional effects have to be included in the model equation. On the other hand, for small amplitudes oscillations it usually turns out that a linear model describes these oscillations sufficiently accurately.

From practical point of view uniform damping is important. In Eq. (8) for $u(x,t)$ the term $\varepsilon a u_t$ (due to den Hartog instability criterion) gives rise to instabilities. After the transformation $u(x,t) = \exp((\varepsilon a t/2)v(x,t))$ an initial boundary value problem (16)–(22) for $v(x,t)$ is obtained. For this problem the damping rates have been determined (see Table 1). In order to have always damping for $u(x,t)$ it easily follows that $a/2 + \mu_1 + 0(\varepsilon)$ has to be smaller than zero for all μ_1. So, to have uniform damping μ_1 has to satisfy $\mu_1 < -a/2$. For the combination of two dampers (damping proportional to the velocity and damping proportional to the angular velocity) it has been proven that damping really occurs in the system and from the numerical results it can be conjectured that in this case a uniform damping can be realized.

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